FLIPS OF MODULI OF STABLE TORSION FREE SHEAVES WITH $c_1 = 1$ ON $\mathbb{P}^2$

RYO OHKAWA

Abstract. We study flips of moduli schemes of stable torsion free sheaves E with $c_1(E) = 1$ on $\mathbb{P}^2$ as wall-crossing phenomena of moduli schemes of stable modules over certain finite dimensional algebra. They are described as stratified Grassmann bundles.

Dedicated to Takao Fujita on the occasion of his 60th birthday

1. Introduction

1.1. **Background.** We denote by $M_{\mathbb{P}^2}(r, c_1, n)$ the moduli of semistable torsion free sheaves E on $\mathbb{P}^2$ with the Chern class $c(E) = (r, c_1, n) \in H^*(\mathbb{P}^2, \mathbb{Z})$. In this paper we treat the case where $c_1 = 1$. In this case semistability and stability for E coincide. When $n \geq r \geq 2$, or $n \geq 2$ and $r = 1$, the Picard number of $M_{\mathbb{P}^2}(r, 1, n)$ is equal to 2 and we have two birational morphisms from $M_{\mathbb{P}^2}(r, 1, n)$, which is described below.

One is defined by J. Li [Li97] for general cases. We denote by $M_{\mathbb{P}^2}(r, 1, n)_0$ the open subset of $M_{\mathbb{P}^2}(r, 1, n)$ consisting of stable vector bundles. The Uhlenbeck compactification $\overline{M}_{\mathbb{P}^2}(r, 1, n)$ of $M_{\mathbb{P}^2}(r, 1, n)_0$ is described set theoretically by

$$\overline{M}_{\mathbb{P}^2}(r, 1, n) = \sqcup_{i \geq 0} M_{\mathbb{P}^2}(r, 1, n-i)_0 \times S^i(\mathbb{P}^2).$$

The map $\pi\colon M_{\mathbb{P}^2}(r, 1, n) \to \overline{M}_{\mathbb{P}^2}(r, 1, n)\colon E \mapsto \pi(E)$ is defined by

$$\pi(E) := (E^{\vee\vee}, \mathrm{Supp}(E^{\vee\vee}/E)) \in M_{\mathbb{P}^2}(r, 1, n-i)_0 \times S^i(\mathbb{P}^2),$$

where $E^{\vee\vee}$ is the double dual of E and i is the length of $E^{\vee\vee}/E$. In the case where $r = 1$, this morphism is called the Hilbert-Chow morphism $\pi\colon (\mathbb{P}^2)^{[n]} \to S^n(\mathbb{P}^2)$ and it is a divisorial contraction when $n \geq 2$. In the case where $r \geq 2$, this map is birational since it is an isomorphism on $M_{\mathbb{P}^2}(r, 1, n)_0$ to its image. It is shown that the codimension of the complement of $M_{\mathbb{P}^2}(r, 1, n)_0$ is equal to 1 when $M_{\mathbb{P}^2}(r, 1, n-1) \neq \emptyset$ (cf. [Mar88, Proposition 3.23]). Hence this map is a divisorial contraction.

The other one is defined by Yoshioka. In his paper [Yos03] on moduli of torsion free sheaves on rational surfaces, he studied the following morphism

$$\psi\colon M_{\mathbb{P}^2}(r, 1, n) \to M_{\mathbb{P}^2}(n+1, 1, n).$$

For any $E \in M_{\mathbb{P}^2}(r, 1, n)$, $\psi(E)$ is defined by the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2})^\vee \otimes \mathcal{O}_{\mathbb{P}^2} \to \psi(E) \to E \to 0, \tag{1}$$

which is called the universal extension, where $\mathrm{Ext}^1_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2})^\vee$ is the dual vector space of $\mathrm{Ext}^1_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2})$. Here we have $\mathrm{Hom}_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2}) = \mathrm{Ext}^2_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2}) = 0$ and $(n+1, 1, n) \in H^*(\mathbb{P}^2, \mathbb{Z})$ is the Chern class of

$$[E] - \chi(E, \mathcal{O}_{\mathbb{P}^2})[\mathcal{O}_{\mathbb{P}^2}] = [E] + \dim \mathrm{Ext}^1_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2})[\mathcal{O}_{\mathbb{P}^2}] \in K(\mathbb{P}^2),$$

where $\chi(E, \mathcal{O}_{\mathbb{P}^2}) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i_{\mathbb{P}^2}(E, \mathcal{O}_{\mathbb{P}^2})$.

Furthermore the moduli space $M_{\mathbb{P}^2}(r,1,n)$ has a stratification

$$M_{\mathbb{P}^2}(r,1,n)=\sqcup_{i=0}^{r}M^i_{\mathbb{P}^2}(r,1,n),$$

where $M^i_{\mathbb{P}^2}(r,1,n):=\{E\in M_{\mathbb{P}^2}(r,1,n)\mid \dim_{\mathbb{C}}\operatorname{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2},E)=i\}$ and it is called the *Brill-Noether locus.* The following theorem is shown in [Yos03].

Theorem 1.1. cf. [Yos03, Theorem 5.8] *The following hold.*

(1) *There exists an isomorphism*

$$M^i_{\mathbb{P}^2}(r,1,n)\cong\psi^{-1}\left(M^{n-r+i+1}_{\mathbb{P}^2}(n+1,1,n)\right).$$

(2) *The restriction of ψ to each strata $M^i_{\mathbb{P}^2}(r,1,n)$ is a $Gr(n-r+i+1,i)$-bundle over the strata $M^{n-r+i+1}_{\mathbb{P}^2}(n+1,1,n)$.*

By the above theorem if n is large enough, ψ is a birational morphism to the image $\operatorname{im}\psi$ and it is a flipping contraction. By the theory of the birational geometry [BCHM10] we have the diagram called flip

$$\begin{tikzcd} M_+(r,1,n) \arrow[dr, "\psi_+"'] & & M_{\mathbb{P}^2}(r,1,n). \arrow[ll, dashed] \arrow[dl, "\psi"] \\ & \operatorname{im}\psi & \end{tikzcd} \tag{2}$$

The purpose of this note is to describe spaces $M_+(r,1,n)$, $\operatorname{im}\psi$ and the morphism ψ_+ in the above diagram using terms of moduli spaces. We follow ideas in [Ohk]. We consider $M_{\mathbb{P}^2}(r,1,n)$ as a moduli scheme of semistable modules over the finite dimensional algebra $\operatorname{End}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1)\oplus\Omega_{\mathbb{P}^2}(3)\oplus\mathcal{O}_{\mathbb{P}^2}(2))$ and study the wall-crossing phenomena as the stability changes using the result of [Ohk] as follows.

1.2. **Main results.** We introduce the exceptional collection

$$\mathfrak{E}:=\left(\mathcal{O}_{\mathbb{P}^2}(1),\Omega^1_{\mathbb{P}^2}(3),\mathcal{O}_{\mathbb{P}^2}(2)\right)$$

on $\mathbb{P}^2$ and put $\mathcal{E}:=\mathcal{O}_{\mathbb{P}^2}(1)\oplus\Omega^1_{\mathbb{P}^2}(3)\oplus\mathcal{O}_{\mathbb{P}^2}(2)$ and $B:=\operatorname{End}_{\mathbb{P}^2}(\mathcal{E})$. We denote abelian categories of coherent sheaves on $\mathbb{P}^2$ and finitely generated right B-modules by $\operatorname{Coh}(\mathbb{P}^2)$ and mod-B respectively. Then by Bondal's Theorem [Bon89], the functor $\Phi:=\mathbf{R}\operatorname{Hom}_{\mathbb{P}^2}(\mathcal{E},-)$ gives an equivalence

$$\Phi\colon D^b(\mathbb{P}^2)\cong D^b(B),$$

where $D^b(\mathbb{P}^2)$ and $D^b(B)$ are the bonded derived categories of $\operatorname{Coh}(\mathbb{P}^2)$ and mod-B respectively. The equivalence Φ also induces an isomorphism $\varphi\colon K(\mathbb{P}^2)\cong K(B)$ between the Grothendieck groups of $\operatorname{Coh}(\mathbb{P}^2)$ and mod-B.

For $\alpha\in K(B)$, we put

$$\alpha^{\perp}:=\{\theta\in\operatorname{Hom}_{\mathbb{Z}}(K(B),\mathbb{R})\mid\theta(\alpha)=0\}.$$

Any $\theta\in\alpha^{\perp}$ defines a stability condition of B-modules E with $[E]=\alpha$. We denote by $M_B(\alpha,\theta)$ the moduli space of θ-semistable B-modules E with $[E]=\alpha$. In particular we take

$$\alpha_r=\alpha_{r,n}:=\varphi(n\mathcal{O}_{\mathbb{P}^2}(-1)[2]+(2n-r+1)\mathcal{O}_{\mathbb{P}^2}[1]+(n-1)\mathcal{O}_{\mathbb{P}^2})\in K(B).$$

Here we omit subscription "n" although α_r depends on n, since we almost always fix n in this paper. There exists a wall-and-chamber structure on $\alpha_r^{\perp}$.

When n is large enough we find two chambers C_-,C_+ and a wall $W_0\subset\alpha_r^{\perp}$ between them such that the following propositions hold (cf. § 3). We put

$$M_-(\alpha_r):=M_B(\alpha_r,\theta_-),\quad M_+(\alpha_r):=M_B(\alpha_r,\theta_+),\quad M_0(\alpha_r):=M_B(\alpha_r,\theta_0)$$

for any $\theta_-\in C_-$, $\theta_+\in C_+$ and $\theta_0\in C_0$.

Proposition 1.2. [Ohk, Main Theorem 1.3 (iii)] *We have an isomorphism*

$$M_{\mathbb{P}^2}(r,1,n) \cong M_-(\alpha_r)\colon E \mapsto \Phi(E[1]).$$

We automatically get the following diagram

$$\begin{tikzcd} M_+(\alpha_r) \arrow[dr, "f_+"] & & M_-(\alpha_r). \arrow[dl, "f_-"'] \\ & M_0(\alpha_r) & \end{tikzcd} \tag{3}$$

By analyzing this diagram we see that diagrams (2) and (3) coincide up to isomorphism. In particular we get the following proposition.

Proposition 1.3. *We have isomorphisms*

(1) $M_0(\alpha_r) \cong \operatorname{im}\psi$ *and*

(2) $M_+(\alpha_r) \cong M_+(r,1,n)$.

Proofs of Proposition 1.3 are given in § 3.1 for (1) and in § 3.4 for (2). Using the B-module $S_0 := \Phi(\mathcal{O}_{\mathbb{P}^2}[1])$ we define the Brill-Noether locus similar to one in Yoshioka's theory,

$$M^i_-(\alpha_r) = \{E \in M_-(\alpha_r) \mid \dim_{\mathbb{C}} \operatorname{Hom}_B(S_0, E) = i\},$$

$$M^i_+(\alpha_r) = \{E \in M_+(\alpha_r) \mid \dim_{\mathbb{C}} \operatorname{Hom}_B(E, S_0) = i\}.$$

Our situation is similar to [NY] and we have our main theorem.

Theorem 1.4. *Assume $n \geq r+2$. Then for each i the following hold.*

(1) *The images $f_+(M^i_+(\alpha_r))$ and $f_-(M^i_-(\alpha_r))$ coincide in $M_0(\alpha_r)$.*

We put $M^i_0(\alpha_r) := f_+(M^i_+(\alpha_r)) = f_-(M^i_-(\alpha_r))$.

(2) *We have isomorphisms $M^i_0(\alpha_r) \cong M^0_-(\alpha_{r-i}) \cong M^0_+(\alpha_{r-i})$.*

(3) *We have isomorphisms $M^i_+(\alpha_r) \cong f_+^{-1}(M^i_0(\alpha_r))$.*

(4) *The restrcition of f_+ to each stratum $M^i_+(\alpha_r) \to M^i_0(\alpha_r)$ is a $Gr(n-r+i-2, i)$-bundle over $M^i_0(\alpha_r)$.*

Note that $M_+(\alpha_r) \neq \emptyset$ if and only if $n \geq r+2$. Proofs of Main Theorem 1.4 are given in § 3.3 for (1) and § 3.6 for the others. We also give a new proof of Theorem 1.1 using terms of B-modules via the isomorphism $M_{\mathbb{P}^2}(r,1,n) \cong M_-(\alpha_r)$ in § 3.6. By these descriptions we see that $M_+(r,1,n)$ is smooth and we can compute Hodge polynomials of $M_+(r,1,n)$ from those of $M_{\mathbb{P}^2}(r,1,n)$.

The paper is organized as follows. In §2 we introduce a description of Picard group of $M_{\mathbb{P}^2}(r,1,n)$ in terms of θ-stability of right B-modules. In §3 we study the wall-crossing phenomena of moduli of θ-semistable right B-modules. This is described as stratified Grassmann bundles and this gives a proof of Main Theorem 1.4. In the Appendix by using Bridgeland stability we give a proof of Proposition 3.11, which is similar to [Ohk, Main Theorem 5.1].

Notation. We fix the following notation in the paper:
If A is a matrix we denote by ${}^tA$ the transpose of A. If V is $\mathbb{C}$-vector space then we denote by $V^\vee$ the dual vector space $\operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})$ of V and we also denote by $Gr(V,i)$ the Grassmann manifold of i-dimensional subspaces of V. We consider the

polynomial ring $\mathbb{C}[x_0, x_1, x_2]$ and the tensor product $V \otimes \mathbb{C}[x_0, x_1, x_2]$with a vector space V. For any monomial $m \in \mathbb{C}[x_0, x_1, x_2]$ we put

$$V \otimes m := \{v \otimes m \in V \otimes \mathbb{C}[x_0, x_1, x_2] \mid v \in V\}.$$

We put $\mathbf{x} := (x_0, x_1, x_2)$ and denote by $V \otimes \mathbf{x}$ the direct sum

$$(V \otimes x_0) \oplus (V \otimes x_1) \oplus (V \otimes x_2)$$

of V. We denote the i-th embedding $V \to V \otimes \mathbf{x}$ and the i-th projection $V \otimes \mathbf{x} \to V$ by x_i and x_i^* for $i = 0, 1, 2$, respectively. For morphisms $f_{ij} : U \to V$ between vector spaces U and V for $i, j = 0, 1, 2$, we denote by the matrix $\begin{pmatrix} f_{00} & f_{01} & f_{02} \\ f_{10} & f_{11} & f_{12} \\ f_{20} & f_{21} & f_{22} \end{pmatrix}$ the morphism

$$f := \sum_{i,j} x_i \circ f_{ij} \circ x_j^* : U \otimes \mathbf{x} \to V \otimes \mathbf{x}.$$

We also use similar notation for vector bundles.

For any path algebra of quiver with relations, we identify modules over the algebra and representations of the corresponding quiver with relations.

2. Picard group of $M_{\mathbb{P}^2}(r, 1, n)$

We introduce an explicit description of the Picard group of $M_{\mathbb{P}^2}(r, 1, n)$ in terms of B-modules.

2.1. **Finite dimensional algebra B.** Finite dimensional algebra $B = \mathrm{End}_{\mathbb{P}^2}(\mathcal{E})$ is written as a path algebra of the following quiver with relations (Q, J), where Q is defined as

$$Q := \overset{v_{-1}}{\bullet} \xleftarrow{\gamma_i} \overset{v_0}{\bullet} \xleftarrow{\delta_j} \overset{v_1}{\bullet}, \ (i, j = 0, 1, 2)$$

and J is generated by the following relations

$$\gamma_i \delta_j + \gamma_j \delta_i = 0, (i, j = 0, 1, 2). \tag{4}$$

We identify categories $D^b(\mathbb{P}^2)$ and $D^b(B)$ and groups $K(\mathbb{P}^2)$ and $K(B)$ via Φ and φ. For example, we denote $\mathcal{O}_{\mathbb{P}^2}(i-1)[2-i]$ and the corresponding simple B-module

$$\mathbb{C}v_i = \Phi(\mathcal{O}_{\mathbb{P}^2}(i-1)[2-i])$$

by the same symbols S_i for $i = -1, 0, 1$.

We put $e_i := [S_i] \in K(B)$ $(i = -1, 0, 1)$. Then we have

$$K(B) = \mathbb{Z}e_{-1} \oplus \mathbb{Z}e_0 \oplus \mathbb{Z}e_1.$$

We denote the dual base by $\{e_{-1}^*, e_0^*, e_1^*\}$. For $\alpha_{-1}, \alpha_0, \alpha_1 \in \mathbb{Z}$, by

$$\alpha = \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{pmatrix} \in K(B) \tag{5}$$

we denote $\alpha = \alpha_{-1}e_{-1} + \alpha_0 e_0 + \alpha_1 e_1 \in K(B)$ and for $\theta^{-1}, \theta^0, \theta^1 \in \mathbb{R}$, by

$$\theta = (\theta^{-1}, \theta^0, \theta^1) \in \mathrm{Hom}_{\mathbb{Z}}(K(B), \mathbb{R}),$$

we denote $\theta = \theta^{-1}e_{-1}^* + \theta^0 e_0^* + \theta^1 e_1^* \in \mathrm{Hom}_{\mathbb{Z}}(K(B), \mathbb{R})$.

2.2. **Moduli of semistable B-modules.** For any $\alpha \in K(B)$ and $\theta \in \alpha^\perp \otimes \mathbb{R} \subset \mathrm{Hom}_{\mathbb{Z}}(K(B), \mathbb{R})$, we define θ-stability as follows. Here

$$\alpha^\perp = \{\theta \in \mathrm{Hom}_{\mathbb{Z}}(K(B), \mathbb{Z}) \mid \theta(\alpha) = 0\}.$$

Definition 2.1. *A right B-module E with $[E] = \alpha$ in $K(B)$ is said to be θ-semistable if for any proper submodule $F \subset E$, the inequality $\theta(F) \geq \theta(E) = 0$ holds. If the inequality is always strict, then E is said to be θ-stable.*

By $M_B(\alpha, \theta)$ we denote a moduli scheme of θ-semistable B-module E with $[E] = \alpha$. We define wall and chamber structure on $\alpha^\perp \otimes \mathbb{R}$ as follows. Wall is a ray $W = \mathbb{R}_{\geq 0}\theta^W$ in $\alpha^\perp \otimes \mathbb{R}$ satisfying that there exists a θ^W-semistable B-module E such that E has a proper submodule F with $[F] \notin \mathbb{Q}_{>0}\alpha$ in $K(B)$ and $\theta^W(F) = 0$. A chamber is a connected component of $(\alpha^\perp \otimes \mathbb{R}) \setminus \cup W$, where W runs over the set of all walls in $\alpha^\perp \otimes \mathbb{R}$. For any chamber $C \subset \alpha^\perp \otimes \mathbb{R}$, the moduli space $M_B(\alpha, \theta)$ does not depend on the choice of $\theta \in C$.

Here we assume that $\alpha \in K(B)$ is indivisible and θ is not on any wall in $\alpha^\perp$, then there exists a universal family $\mathcal{U}$ of B-modules on $M_B(\alpha, \theta)$

$$\mathcal{U} := \left(\mathcal{U}_{-1} \xrightarrow{\quad \gamma_i^* \quad} \mathcal{U}_0 \xrightarrow{\quad \delta_j^* \quad} \mathcal{U}_1\right), (i, j = 0, 1, 2) \tag{6}$$

where $\mathcal{U}_{-1}$, $\mathcal{U}_0$ and $\mathcal{U}_1$ are vector bundles corresponding to vertices v_{-1}, v_0, v_1 and $\gamma_i^* \colon \mathcal{U}_{-1} \to \mathcal{U}_0$, $\delta_j^* \colon \mathcal{U}_0 \to \mathcal{U}_1$ are morphisms corresponding to arrows γ_i, δ_j.

2.3. **Deformations of B-modules.** We take $\alpha \in K(B)$ defined by (5). For any B-module E with $[E] = \alpha$, by choosing basis of Ev_{-1}, Ev_0 and Ev_1 we have an isomorphism

$$E \cong (\mathbb{C}^{\alpha_{-1}} \overset{C_i}{\to} \mathbb{C}^{\alpha_0} \overset{D_j}{\to} \mathbb{C}^{\alpha_1}), \tag{7}$$

where $C_i \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_0})$ and $D_j \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_1})$ correspond to the action of γ_i and δ_j respectively for $i, j = 0, 1, 2$. The pull back of the heart mod-B of the standard t-structure of $D^b(B)$ by Φ is a full subcategory $\mathcal{A} := \langle \mathcal{O}(-1)[2], \mathcal{O}[1], \mathcal{O}(1)\rangle$ of $D^b(\mathbb{P}^2)$. The following complex of coherent sheaves on $\mathbb{P}^2$

$$\mathcal{O}(-1)^{\alpha_{-1}} \xrightarrow{\sum_i C_i x_i} \mathcal{O}^{\alpha_0} \xrightarrow{\sum_i D_j x_j} \mathcal{O}(1)^{\alpha_1}$$

corresponds to E in (7) via the equivalence Φ, where x_0, x_1, x_2 are homogeneous coordinates of $\mathbb{P}^2$. By [Ohk, Lemma 4.6 (1)], $\mathrm{Ext}^2_B(E, E)$ is isomorphic to the cokernel of the map

$$\left(\bigoplus_i \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_0})x_i\right) \bigoplus \left(\bigoplus_j \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_1})x_j\right) \overset{d}{\to} \bigoplus_{i \leq j} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_1})x_i x_j, \tag{8}$$

where the map d is defined by

$$d(\sum_i \xi_i x_i, \sum_j \eta_j x_j) = \sum_{i,j}(D_j \xi_i + \eta_i C_j)x_i x_j$$

for $\xi_i \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_0})$ and $\eta_j \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_1})$, $(i, j = 0, 1, 2)$. We study the deformation functor $\mathcal{D}_E \colon (\mathrm{Artin}/k) \to (\mathrm{Sets})$. For any Artin local k-ring R, the set $\mathcal{D}_E(R)$ consists of right $R \otimes B$-modules E^R

$$E^R = (R^{\alpha_{-1}} \overset{C_i^R}{\to} R^{\alpha_0} \overset{D_j^R}{\to} R^{\alpha_1}), D_j^R C_i^R + D_i^R C_j^R = 0$$

such that $C_i^R \equiv C_i, D_j^R \equiv D_j$ modulo m_R for each $i, j = 0, 1, 2$, where C_i^R and D_j^R is R-linear maps and m_R is the maximal ideal of R. We show the following lemma.

Lemma 2.2. *The deformation functor $\mathcal{D}_E$ has an obstruction theory with values in* $\mathrm{Ext}^2_B(E,E)$.

Proof. For any small extension

$$0 \to \mathfrak{a} \to R' \to R \to 0$$

with $m_R\mathfrak{a} = 0$ and $E^R = (C_i^R, D_j^R) \in \mathcal{D}_E(R)$, we write $C_i^R = C_i + \xi_i$ and $D_j^R = D_j + \eta_j$ for $\xi_i \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_0}) \otimes m_R$ and $\eta_j \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_1}) \otimes m_R$. By the isomorphism $m_R \cong m_{R'}/\mathfrak{a}$, we have lifts $\xi_i' \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_0}) \otimes m_{R'}$ and $\eta_j' \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_0}, \mathbb{C}^{\alpha_1}) \otimes m_{R'}$ of ξ_i and η_j respectively. We put $C_j^{R'} := C_j + \xi_i'$, $D_j^{R'} := D_j + \eta_j'$.

Since $D_j^{R'}C_i^{R'} + D_i^{R'}C_j^{R'} \equiv D_j^R C_i^R + D_i^R C_j^R = 0$ modulo $\mathfrak{a}$, we have an element

$$\sum_{i \le j} \left(D_j^{R'}C_i^{R'} + D_i^{R'}C_j^{R'} \right) x_i x_j \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{-1}}, \mathbb{C}^{\alpha_1}) x_i x_j \otimes \mathfrak{a}.$$

By the image of this element to the cokernel of (8) tensored by $\mathfrak{a}$, we define an element $\mathfrak{o}(E^R)$ in $\mathrm{Ext}^2_B(E,E) \otimes \mathfrak{a}$. This defines a well-defined map $\mathfrak{o} \colon \mathcal{D}_E(R) \to \mathrm{Ext}^2_B(E,E) \otimes \mathfrak{a}$ and we easily see that E^R lifts to $\mathcal{D}_E(R')$ if and only if $\mathfrak{o}(E^R) = 0$. □

2.4. **Chambers $C_{\mathbb{P}^2}$ and Picard group of $M_{\mathbb{P}^2}(r,1,n)$.** We take

$$\alpha_r = \begin{pmatrix} n \\ 2n + r - 1 \\ n - 1 \end{pmatrix} \in K(B) \tag{9}$$

such that $\mathrm{ch}(\alpha_r) = -(r, 1, \frac{1}{2} - n)$ and assume that $M_{\mathbb{P}^2}(r,1,n)$ is not empty set. Then there exists a chamber $C_{\mathbb{P}^2} \subset \alpha_r^{\perp} \otimes \mathbb{R}$ such that $\Phi(\,\cdot\,[1])$ induces an isomorphism

$$M_{\mathbb{P}^2}(r,1,n) \cong M_B(\alpha_r, \theta) \tag{10}$$

for any $\theta \in C_{\mathbb{P}^2}$ (see [Ohk, Main Theorem 5.1] or [Pot94]). The chamber $C_{\mathbb{P}^2}$ is characterized as follows. We put $\theta_{\mathbb{P}^2} := (-r-1, 1, -1+r)$. Then we have $\theta_{\mathbb{P}^2}(\mathcal{O}_x) = 0$ for any skyscraper sheaf at $x \in \mathbb{P}^2$. The closure of $C_{\mathbb{P}^2}$ contains the ray $\mathbb{R}_{\ge 0}\theta_{\mathbb{P}^2}$ and for certain $\theta \in C_{\mathbb{P}^2}$ we have $\theta(\mathcal{O}_x) > 0$ and $M_B(\alpha_r, \theta) \neq \emptyset$.

If we put $\theta_0 := (-n+1, 0, n)$, then by [Ohk, Lemma 6.2] we have

$$\mathbb{R}_{>0}\theta_0 + \mathbb{R}_{>0}\theta_{\mathbb{P}^2} \subset C_{\mathbb{P}^2}. \tag{11}$$

In the case where $r = 1$, by [Ohk, Lemma 6.3 (2)] we have $\mathbb{R}_{>0}\theta_0 + \mathbb{R}_{>0}\theta_{\mathbb{P}^2} = C_{\mathbb{P}^2}$ for $n \ge 2$ In the case where $r \ge 2$, we will describe the chamber in § 3.4.

Since α_r is indivisible by the definition (9), for any $\theta \in C_{\mathbb{P}^2}$ we have a universal family $\mathcal{U}$ on $M_B(\alpha_r, \theta)$ as in (6). We define a homomorphism from $\alpha_r^{\perp}$ to $\mathrm{Pic}\,(M_B(\alpha_r, \theta))$ by

$$\rho(\mathbf{m}) = m_{-1} \det(\mathcal{U}_{-1}) + m_0 \det(\mathcal{U}_0) + m_1 \det(\mathcal{U}_1),$$

for $\mathbf{m} = (m_{-1}, m_0, m_1) \in \alpha_r^{\perp}$. By (10) this gives a homomorphism $\rho \colon \alpha_r^{\perp} \to \mathrm{Pic}\,(M_{\mathbb{P}^2}(r,1,n))$. Then by [Dre98] we have the following proposition.

Proposition 2.3. *The above map $\rho \colon \alpha_r^{\perp} \to \mathrm{Pic}\,(M_{\mathbb{P}^2}(r,1,n))$ is an isomorphism. Furthermore $\rho(-3\theta_{\mathbb{P}^2})$ is the canonical bundle of $M_{\mathbb{P}^2}(r,1,n)$.*

3. Proof of Main Theorem 1.4

We put $\alpha_{r,n} := {}^t(n, 2n-1+r, n-1) \in K(B)$. In the following we omit "n" and put $\alpha_r = \alpha_{r,n}$ except in § 3.4. Note that $\mathrm{ch}(\alpha_r) = -(r, 1, \frac{1}{2} - n)$. We put $\theta_0 := (-n+1, 0, n) \in \alpha_r^{\perp}$ and consider $\theta_+ := \theta_0 + \varepsilon(2n-1+r, -n, 0)$ and $\theta_- := \theta_0 - \varepsilon(2n-1+r, -n, 0)$ for $\varepsilon > 0$ small enough such that $\theta_\pm$ lie on no wall. We put $M_\pm(\alpha_r) := M_B(\alpha_r, \theta_\pm)$ and $M_0(\alpha_r) := M_B(\alpha_r, \theta_0)$. By (10) and (11) we have an isomorphism

$$M_{\mathbb{P}^2}(r, 1, n) \cong M_-(\alpha_r). \tag{12}$$

By $C_\pm$ we denote the chamber containing $\theta_\pm$ respectively and put $W_0 := \mathbb{R}_{\geq 0}\theta_0$. Since $\theta_- \in C_{\mathbb{P}^2}$, we have $C_- = C_{\mathbb{P}^2}$. We automatically get the following diagram:

$$\begin{tikzcd} M_+(\alpha_r) \arrow[dr, "f_+"] & & M_-(\alpha_r). \arrow[dl, "f_-"'] \\ & M_0(\alpha_r) & \end{tikzcd} \tag{13}$$

In this section we see that this diagram is described by stratified Grassmann bundles and give a proof of Main Theorem 1.4.

3.1. **Kronecker modules.** We consider the 3-Kronecker quiver, which has 2 vertices v_{-1}, v_1 and 3 arrows $\beta_0, \beta_1, \beta_2$ from v_1 to v_{-1}

$$\overset{v_{-1}}{\bullet} \xleftarrow[\beta_i]{\quad\quad} \overset{v_1}{\bullet}\ , (i = 0, 1, 2).$$

and consider the path algebra T. Any right T-module G has a decomposition $G = Gv_{-1} \oplus Gv_1$ and actions of β_i define linear maps $Gv_{-1} \to Gv_1$ for $i = 0, 1, 2$. By abbreviation we define $\theta_0(G) \in \mathbb{R}$ by

$$\theta_0(G) := (-n+1)\dim_{\mathbb{C}} Gv_{-1} + n \dim_{\mathbb{C}} Gv_1.$$

We denote by $K(T)$ the Grothendieck group of the abelian category of finitely generated right T-modules and take $\alpha_T := n[\mathbb{C}v_{-1}] + (n-1)[\mathbb{C}v_1] \in K(T)$.

Definition 3.1. *An right T-module G with $[G] = \alpha_T \in K(T)$ is stable if and only if for any non-zero proper submodule G' of G we have an inequality $\theta_0(G') > 0$.*

We denote by $M_T(\alpha_T)$ the moduli space of stable T-modules G with $[G] = \alpha_T$. For any B-module $E = \left(\mathbb{C}^n \overset{C_i}{\to} \mathbb{C}^{2n-1+r} \overset{D_j}{\to} \mathbb{C}^{n-1}\right)$, we define T-module E_T by

$$E_T := \left(\mathbb{C}^n \overset{A_i}{\to} \mathbb{C}^{n-1}\right),$$

where C_i and D_j are matrices with suitable sizes and we define A_i by $A_i := D_{i+2}C_{i+1}$ for each $i \in \mathbb{Z}/3\mathbb{Z}$.

Lemma 3.2. *For any B-module $E = \left(\mathbb{C}^n \overset{C_i}{\to} \mathbb{C}^{2n-1+r} \overset{D_j}{\to} \mathbb{C}^{n-1}\right)$, the following hold.*

(1) *E is θ_0-semistable if and only if E_T is stable.*
(2) *E is θ_0-stable if and only if E_T is stable and*

$$\mathrm{Hom}_B(E, S_0) = \mathrm{Hom}_B(S_0, E) = 0.$$

(3) *The following are equivalent.*
  (a$-$) *E is θ_--stable.*
  (b$-$) *E is θ_--semistable.*
  (c$-$) *E_T is stable and $\mathrm{Hom}_B(E, S_0) = 0$.*
(4) *The following are equivalent.*

(a+) *E is θ_+-stable.*
(b+) *E is θ_+-semistable.*
(c+) *E_T is stable and* $\mathrm{Hom}_B(S_0, E) = 0$.

Proof. (1) For every submodule $F \subset E$, we have a submodule F_T of E_T. Conversely for any submodule G' of E_T, we define a submodule F of E such that $F_T = G'$ as follows. We put $Fv_{-1} := G'v_{-1}$, $Fv_1 := G'v_1$ and $Fv_0 := \sum_i C_i(Fv_{-1}) \subset Ev_0$. By the relations (4) we have a submodule $F := Fv_{-1} \oplus Fv_0 \oplus Fv_1$ of E and $\theta_0(F) = \theta_0(F_T) = \theta_0(G')$. This yields the claim.
(2) For any non-zero proper submodule $F \subset E$, $\theta_0(F) = 0$ if and only if $\underline{\dim}(F) = (0, l, 0)$ or $(n, l, n-1)$ for $0 < l < 2n + r - 1$. There exists no such F if and only if $\mathrm{Hom}_B(S_0, E) = \mathrm{Hom}_B(E, S_0) = 0$.
(3) (a−) $\Longrightarrow$ (b−) It is trivial. (b−) $\Longrightarrow$ (c−) We choose $\theta_- = \theta_0 - \varepsilon(2n - 1 + r, -n, 0)$ for $\varepsilon > 0$ small enough. If E is θ_--semistable, then for any submodule $F \subset E$ we have $\theta_0(F) \geq 0$, since $\theta_-(F) \geq 0$ for arbitrary small $\varepsilon > 0$. This implies that E is θ_0-semistable and hence by (1), E_T is semistable. Any non-zero $\phi \in \mathrm{Hom}_B(E, S_0)$ destabilize E. Hence we also have $\mathrm{Hom}_B(E, S_0) = 0$.
(c−) $\Longrightarrow$ (a−) We assume that E_T is stable and $\mathrm{Hom}_B(E, S_0) = 0$. Hence for every submodule $F \subset E$, we have $\theta_0(F) = \theta_0(F_T) \geq 0$. If $\theta_0(F) = 0$ then $\mathrm{Hom}_B(E, S_0) = 0$ implies that $F \cong S_0^{\oplus l}$ for $0 \leq l \leq 2n + r - 1$. In this case $\theta_-(F) = \varepsilon n l > 0$. If $\theta_0(F) > 0$ then we also have $\theta_-(F) > 0$ for ε small enough. Hence E is θ_--stable.
(4) It is similar to the proof of (3).

□

By the above lemma we have morphisms $\pi^r_\pm \colon M_\pm(\alpha_r) \to M_T(\alpha_T) \colon E \mapsto E_T$. We also see that the map $E \mapsto E_T$ is independent of representatives of S-equivalence class for θ_0-stability up to isomorphism of T-modules. Hence we get the morphism $\pi^r_0 \colon M_0(\alpha_r) \to M_T(\alpha_T)$ and this map is set theoretically injective.

Lemma 3.3. *The morphism $\pi^r_0 \colon M_0(\alpha_r) \to M_T(\alpha_T)$ gives a closed embedding.*

Proof. π^r_0 is induced from a homomorphism of graded rings of invariant sections, therefore affine morphism. Since both of $M_0(\alpha_r)$ and $M_T(\alpha_T)$ are projective, π^r_0 is finite. Since π^r_0 is set theoretically injective, the claim holds. □

Furthermore we easily see that morphisms

$$\pi^{n+1}_- \colon M_-(\alpha_{n+1}) \to M_T(\alpha_T), \pi^{n-2}_+ \colon M_+(\alpha_{n-2}) \to M_T(\alpha_T)$$

are isomorphisms. Inverse maps

$$E_T = \left(\mathbb{C}^n \overset{A_i}{\to} \mathbb{C}^{n-1}\right) \mapsto E_\pm = \left(\mathbb{C}^n \overset{C_i^\pm}{\to} \mathbb{C}^{2n-1+r} \overset{D_i^\pm}{\to} \mathbb{C}^{n-1}\right)$$

of π^{n-2}_+ and π^{n+1}_- are defined by

$$(C_0^+, C_1^+, C_2^+) = \begin{pmatrix} 0 & -A_2 & A_1 \\ A_2 & 0 & -A_0 \\ -A_1 & A_0 & 0 \end{pmatrix}, \begin{pmatrix} D_0^+ \\ D_1^+ \\ D_2^+ \end{pmatrix} = I_{3n-3} \text{ for } \pi^{n-2}_+$$

and

$$(C_0^-, C_1^-, C_2^-) := I_{3n}, \begin{pmatrix} D_0^- \\ D_1^- \\ D_2^- \end{pmatrix} := \begin{pmatrix} 0 & -A_2 & A_1 \\ A_2 & 0 & -A_0 \\ -A_1 & A_0 & 0 \end{pmatrix} \text{ for } \pi^{n+1}_-,$$

where I_{3n} and I_{3n-3} are unit matrices with sizes $3n$ and $3n - 3$ respectively.

Hence we get the diagram:

$$
\begin{tikzcd}
M_+(\alpha_r) \arrow[dr, "f_+"] \arrow[dd, "g_+"'] & & M_-(\alpha_r) \arrow[dl, "f_-"'] \arrow[dd, "g_-"] \\
& M_0(\alpha_r) \arrow[d, "\pi_0^r"] & \\
M_+(\alpha_{n-2}) \arrow[r, "\cong", "\pi_+^{n-2}"'] & M_T(\alpha_T) & M_-(\alpha_{n+1}) \arrow[l, "\cong"', "\pi_-^{n+1}"]
\end{tikzcd}
\tag{14}
$$

where $g_- := (\pi_-^{n+1})^{-1} \circ \pi_0^r \circ f_-$ and $g_+ := (\pi_+^{n-2})^{-1} \circ \pi_0^r \circ f_+$. Morphisms g_- and g_+ are explicitly defined by the following universal extensions for each $E_- \in M_-(\alpha_r)$ and $E_+ \in M_+(\alpha_r)$,

$$0 \to \operatorname{Ext}^1_B(E_-, S_0)^\vee \otimes S_0 \to g_-(E_-) \to E_- \to 0, \tag{15}$$

$$0 \to E_+ \to g_+(E_+) \to \operatorname{Ext}^1_B(S_0, E_+) \otimes S_0 \to 0. \tag{16}$$

Hence via isomorphisms (12), the morphism g_- coincides with Yoshioka's map ψ, which is defined by a similar exact sequence (1). By Lemma 3.2 and the diagram (14), we have

$$M_0(\alpha_r) \cong \operatorname{im} g_- \cong \operatorname{im} \psi.$$

This gives a proof of (1) in Proposition 1.3.

3.2. **Brill-Noether locus.** We introduce the Brill-Noether locus $M^i_-(\alpha_r)$ and $M^i_+(\alpha_r)$ as follows.

$$M^i_-(\alpha_r) := \{E_- \in M_-(\alpha_r) \mid \dim_{\mathbb{C}} \operatorname{Hom}_B(S_0, E_-) = i\},$$

$$M^i_+(\alpha_r) := \{E_+ \in M_+(\alpha_r) \mid \dim_{\mathbb{C}} \operatorname{Hom}_B(E_+, S_0) = i\}.$$

When we replace '=i' by '$\geq$ i' in the right hand side, the corresponding moduli spaces are denoted by the left hand side with 'i' replaced by '$\geq$ i'.

If we put $\delta^* := \sum_i x_i \circ \delta_i^* \colon \mathcal{U}_0 \to \mathcal{U}_1 \otimes \mathbf{x}$, then the zero locus of $\wedge^{\operatorname{rk}\mathcal{U}_0 - i+1}\delta^*$ defines $M^{\geq i}_-(\alpha_r)$ as a closed subscheme of $M_-(\alpha_r)$ because $\ker \delta^*_x \cong \operatorname{Hom}_B(S_0, \mathcal{U}_x)$ for any $x \in M_-(\alpha_r)$, where $\mathcal{U} = \left(\mathcal{U}_{-1} \overset{\gamma_i^*}{\to} \mathcal{U}_0 \overset{\delta_j^*}{\to} \mathcal{U}_1\right)$ is a universal family of B-modules on $M_-(\alpha_r)$. Similarly, $M^{\geq i}_+(\alpha_r)$ is defined as a closed subscheme of $M_+(\alpha_r)$. $M^i_-(\alpha_r) = M^{\geq i}_-(\alpha_r) \setminus M^{\geq i+1}_-(\alpha_r)$ and $M^i_+(\alpha_r) = M^{\geq i}_+(\alpha_r) \setminus M^{\geq i+1}_+(\alpha_r)$ are open subset of $M^{\geq i}_-(\alpha_r)$ and $M^{\geq i}_+(\alpha_r)$, respectively.

3.3. **Set-theoretical description of Grassmann bundles.** By Lemma 3.2 we have the following proposition.

Proposition 3.4. *The following hold.*

(1) *For any $E_- \in M^i_-(\alpha_r)$, we put $E' := \operatorname{coker}(\operatorname{Hom}_B(S_0, E_-) \otimes S_0 \to E_-)$. Then E' is θ_--semistable and* $\operatorname{Hom}_B(S_0, E') = 0$*, that is, $E' \in M^0_-(\alpha_{r-i})$. Hence E' is also θ_0-stable.*

(2) *Conversely, for any $E' \in M^0_-(\alpha_{r-i})$ and any i-dimensional vector subspace $V \subset \operatorname{Ext}^1_B(E', S_0)$, we obtain a B-module E_- by the canonical exact sequence*

$$0 \to V^\vee \otimes S_0 \to E_- \to E' \to 0.$$

Then E_- is θ_--semistable and $\operatorname{Hom}_B(S_0, E_-) \cong V$*, that is, $E_- \in M^i_-(\alpha_r)$.*

(3) *For any $E_+ \in M^i_+(\alpha_r)$, we put $E' := \ker(E_+ \to \operatorname{Hom}_B(E, S_0)^\vee \otimes S_0)$. Then E' is θ_+-semistable and* $\operatorname{Hom}_B(E', S_0) = 0$*, that is, $E' \in M^0_+(\alpha_{r-i})$. Hence E' is also θ_0-stable.*

(4) *Conversely, for any* $E' \in M^0_+(\alpha_{r-i})$ *and any* i*-dimensional vector subspace* $V \subset \mathrm{Ext}^1_B(S_0, E')$*, we obtain a* B*-module* E_+ *by the canonical exact sequence*

$$0 \to E' \to E_+ \to V \otimes S_0 \to 0.$$

Then E_+ *is* θ_+*-semistable and* $\mathrm{Hom}_B(E_+, S_0) \cong V$*, that is,* $E_+ \in M^i_+(\alpha_r)$.

By Lemma 3.2, $M^0_-(\alpha_{r-i})$ is set theoretically equal to $M^0_+(\alpha_{r-i})$. For any B-module E, we have $\mathrm{Ext}^2_B(S_0, E) = \mathrm{Ext}^2_B(E, S_0) = 0$ by [Ohk, Lemma 4.6 (1)]. Hence by the Rieman-Roch formula, for any element $E' \in M^0_-(\alpha_{r-i}) = M^0_+(\alpha_{r-i})$ we have $\dim_{\mathbb{C}} \mathrm{Ext}^1_B(E', S_0) = n + 1 - r + i$ and $\dim_{\mathbb{C}} \mathrm{Ext}^1_B(S_0, E') = n - 2 - r + i$. If $n - 2 - r \geq 0$, then by the above lemma we have set theoretical equalities

$$\begin{aligned} f_-\left(M^i_-(\alpha_r)\right) &= \left\{S_0^{\oplus i} \oplus E' \mid E' \in M^0_-(\alpha_{r-i}) = M^0_+(\alpha_{r-i})\right\} / \equiv_S \\ &= f_+\left(M^i_+(\alpha_r)\right), \end{aligned} \tag{17}$$

where $\equiv_S$ denotes the S-equivalence relation (cf. [Ohk, § 4.1]). This gives a proof of (1) of Main Theorem 1.4. Fibers of S-equivalence class of $S_0^{\oplus i} \oplus E'$ by f_- and f_+ are parametrized by $Gr(\mathrm{Ext}^1_B(E', S_0), i)$ and $Gr(\mathrm{Ext}^1_B(S_0, E'), i)$ for $E' \in M^0_-(\alpha_{r-i}) = M^0_+(\alpha_{r-i})$.

Lemma 3.5. *For any integer* $i > r$ *the following holds.*

$$M^i_-(\alpha_r) = M^i_+(\alpha_r) = \emptyset$$

Proof. By [Yos03, Lemma 5.7], we have $M^i_-(\alpha_r) = \emptyset$ for any $i > r$. By (17) this implies $M^i_+(\alpha_r) = \emptyset$. □

3.4. **Description of $C_{\mathbb{P}^2}$.** In the following proposition we use the symbol $\alpha_{r,n} = {}^t(n, 2n - 1 + r, n - 1) \in K(B)$.

Proposition 3.6. *The following hold.*

(1) $M_{\mathbb{P}^2}(r, 1, n) \neq \emptyset$ *if and only if* $n \geq r - 1$.

In the following, we assume $r \geq 2$.

(2) $W_0 = \mathbb{R}_{\geq 0}\theta_0$ *is a wall on* $\alpha^\perp_{r,n} \otimes \mathbb{R}$ *for* $n \geq r - 1$.
(3) $W_{\mathbb{P}^2} = \mathbb{R}_{\geq 0}\theta_{\mathbb{P}^2}$ *is a wall on* $\alpha^\perp_{r,n} \otimes \mathbb{R}$ *for* $n \geq r$.

Hence we have $\mathbb{R}_{>0}\theta_0 + \mathbb{R}_{>0}\theta_{\mathbb{P}^2} = C_{\mathbb{P}^2}$ *if* $n \geq r$.

Proof. (1) By the criterion for the existence of non exceptional stable sheaves in [Pot97, § 16.4], we have our claim.
(2) We assume $n \geq r - 1$. By (1), there exists an element E of $M_-(\alpha_{r-1,n}) \cong M_{\mathbb{P}^2}(r - 1, 1, n)$. By Lemma 3.2 (1), a B-module $E \oplus S_0$ is θ_0-semistable and has a submodule S_0 with $\theta_0(S_0) = 0$. Hence $W_0 = \mathbb{R}_{\geq 0}\theta_0$ is a wall on $\alpha^\perp_{r,n} \otimes \mathbb{R}$.
(3) We assume $n \geq r$ and take an element $\mathcal{F}$ of $M_{\mathbb{P}^2}(r, 1, n - 1)$. We consider the exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{O}_x \to 0$$

for skyscraper sheaf $\mathcal{O}_x$ at any point $x \in \mathbb{P}^2$. Then $\mathcal{F}'$ is stable since $\mu(\mathcal{F}') = \mu(\mathcal{F})$. This gives elements $F := \Phi(\mathcal{F}[1])$, $F' := \Phi(\mathcal{F}'[1])$ of $M_-(\alpha_{r,n-1})$, $M_-(\alpha_{r,n})$ respectively and an exact sequence of B-modules

$$0 \to \Phi(\mathcal{O}_x) \to F' \to F \to 0.$$

Hence $W_{\mathbb{P}^2} = \mathbb{R}_{\geq 0}\theta_{\mathbb{P}^2}$ is a wall on $\alpha^\perp_r \otimes \mathbb{R}$. These together with (11) imply the last assertion. □

By this proposition and [Ohk, Lemma 6.3 (2)] we have $\mathbb{R}_{>0}\theta_0 + \mathbb{R}_{>0}\theta_{\mathbb{P}^2} = C_{\mathbb{P}^2}$ if $r = 1$ and $n \geq 2$, or $r \geq 2$ and $n \geq r$. In this case C_+ is different from $C_- = C_{\mathbb{P}^2}$ and it is adjacent to $C_- = C_{\mathbb{P}^2}$ with the boundary containing W_0. By the description of the canonical bundle of $M_-(\alpha_r)$ in Proposition 2.3 we see that the diagram (13) gives the flip of $M_-(\alpha_r)$. Hence we get an isomorphism $M_+(\alpha_r) \cong M_+(r, 1, n)$ and a proof of (2) in Proposition 1.3.

Proposition 3.7. *Moduli schemes $M_-(\alpha_r)$ and $M_+(\alpha_r)$ are smooth.*

Proof. By Lemma 2.2, deformation functors of B-modules E have obstruction theories with values in $\mathrm{Ext}^2_B(E, E)$. Since $\mathrm{Ext}^2_B(E_-, E_-) = 0$ for any $E_- \in M_-(\alpha_r) \cong M_{\mathbb{P}^2}(r, 1, n)$ we see that $M_-(\alpha_r)$ is smooth (cf. [HL97, Corollary 4.5.2]). Furthermore if E_+ is an element of $M_+(\alpha_r)$ then by Proposition 3.4 we have an exact sequence

$$0 \to E' \to E_+ \to \mathbb{C}^i \otimes S_0 \to 0$$

for some i and $E' \in M^0_-(\alpha_{r-i})$. Since

$$\mathrm{Ext}^2_B(S_0, E_+) = \mathrm{Ext}^2_B(E', E') = \mathrm{Ext}^2_B(E', S_0) = 0,$$

we also have $\mathrm{Ext}^2_B(E_+, E_+) = 0$. Thus $M_+(\alpha_r)$ is also smooth. □

In the rest of this section we show that the diagram (13) is scheme theoretically described by stratified Grassmann bundles.

3.5. **Coherent systems.** For $r \geq i \geq 0$ we define moduli of coherent systems $M_-(\alpha_r, i)$ and $M_+(\alpha_r, i)$:

$$M_-(\alpha_r, i) := \{(E_-, V) \mid E_- \in M_-(\alpha_r), V \subset \mathrm{Hom}_B(S_0, E_-) \text{ with } \dim_{\mathbb{C}} V = i\},$$

$$M_+(\alpha_r, i) := \{(E_+, V) \mid E_+ \in M_+(\alpha_r), V \subset \mathrm{Hom}_B(E_+, S_0) \text{ with } \dim_{\mathbb{C}} V = i\}.$$

These moduli schemes are constructed as follows. We only show the construction of $M_-(\alpha_r, i)$ because the construction of $M_+(\alpha_r, i)$ is similar.

We introduce the following quiver with relations $(\bar{Q}, I)$, where

$$\bar{Q} := \begin{tikzcd} & u \arrow[d, "\rho"] & \\ \bar{v}_{-1} & \bar{v}_0 \arrow[l, "\bar{\gamma}_i"'] \arrow[d, "\iota"] & \bar{v}_1 \arrow[l, "\bar{\delta}_j"'] \\ & w & \end{tikzcd}, \qquad (i, j = 0, 1, 2)$$

and I is generated by the following relations

$$\bar{\gamma}_i\rho = \iota\bar{\delta}_j = \bar{\gamma}_i\delta_j + \bar{\gamma}_j\delta_i = \iota\rho = 0, (i, j = 0, 1, 2).$$

Let $\bar{B}$ be a path algebra $\mathbb{C}\bar{Q}/I$ of the quiver with relations $(\bar{Q}, I)$. We have simple modules $\mathbb{C}\bar{v}_{-1}$, $\mathbb{C}\bar{v}_0$, $\mathbb{C}\bar{v}_1$, $\mathbb{C}u$ and $\mathbb{C}w$. For each $\alpha_r \in K(B)$, we put

$$\bar{\alpha}_r := n[\mathbb{C}\bar{v}_{-1}] + (2n + r - 1)[\mathbb{C}\bar{v}_0] + (n - 1)[\mathbb{C}\bar{v}_1] + (2n + r - i)[\mathbb{C}u] + i[\mathbb{C}w] \in K(\bar{B}),$$

and for $\theta_- = (\theta^{-1}_-, \theta^0_-, \theta^1_-) \in \alpha_r^{\perp}$ and $\varepsilon' > 0$ small enough, we put

$$\bar{\theta}_- := \theta^{-1}_-[\mathbb{C}\bar{v}_{-1}]^* + \theta^0_-[\mathbb{C}\bar{v}_0]^* + \theta^1_-[\mathbb{C}\bar{v}_1]^* + \frac{\varepsilon'}{2n + r - i - 1}[\mathbb{C}u]^* - \frac{\varepsilon'}{i}[\mathbb{C}w]^* \in \bar{\alpha}_r^{\perp}.$$

For any right $\bar{B}$-module $\bar{E}_-$ with $[\bar{E}_-] = \bar{\alpha}_r \in K(\bar{B})$,

$$\bar{E}_- = \begin{tikzcd} 0 \arrow[r] & \mathbb{C}^{2n-1+r-i} \arrow[r] & 0 \\ \mathbb{C}^n \arrow[u] \arrow[r, "\bar{C}_i"] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{C}^{2n-1+r} \arrow[u, "B"'] \arrow[r, "\bar{D}_j"] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{C}^{n-1} \arrow[u] \\ 0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{C}^i \arrow[u, "A"'] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & 0 \arrow[u] \end{tikzcd}$$

we put

$$E_- := \left(\mathbb{C}^n \xrightarrow{\bar{C}_i} \mathbb{C}^{2n-1+r} \xrightarrow{\bar{D}_j} \mathbb{C}^{n-1} \right).$$

The following lemma is proved similarly as in Lemma 3.2 (3).

Lemma 3.8. *If we take ε' small enough, then $\bar{E}_-$ is $\bar{\theta}_-$-semistable if and only if E_- is θ_--semistable and A is injective and B is surjective.*

Hence if we denote by $M_{\bar{B}}(\bar{\alpha}_r, \bar{\theta}_-)$ the moduli of $\bar{\theta}_-$-semistable $\bar{B}$-module $\bar{E}_-$ with $[\bar{E}_-] = \bar{\alpha}_r$, we get an isomorphism $M_{\bar{B}}(\bar{\alpha}_r, \bar{\theta}_-) \cong M_-(\alpha_r, i)$. We write as $\bar{E}_- = (E_-, \mathbb{C}^i) \in M_-(\alpha_r, i)$ by abbreviation.

We have morphisms

$$q_1 \colon M_-(\alpha_r, i) \to M_-(\alpha_r) \colon \bar{E}_- = (E_-, \mathbb{C}^i) \mapsto E_-$$

and

$$q_2 \colon M_-(\alpha_r, i) \to M_-(\alpha_{r-i}) \colon \bar{E}_- \mapsto q_2(\bar{E}_-)$$

defined by the canonical exact sequence

$$0 \to \mathbb{C}^i \otimes S_0 \to E_- \to q_2(\bar{E}_-) \to 0.$$

Similarly we have morphisms $q_1' \colon M_+(\alpha_r, i) \to M_+(\alpha_r)$ and $q_2' \colon M_+(\alpha_r, i) \to M_+(\alpha_{r-i})$. If we take an element $\bar{E}_+ := (E_+, \mathbb{C}^i) \in M_+(\alpha_r, i)$, then q_1' and q_2' are defined by $q_1'(\bar{E}_+) = E_+$ and $q_2'(\bar{E}_+) := \ker\big(E_+ \to (\mathbb{C}^i)^* \otimes S_0\big)$.

Proposition 3.9. *The following hold.*

(1) *The morphism $q_1 \colon M_-(\alpha_r, i) \to M_-(\alpha_r)$ is a $Gr(j,i)$-bundle over each strata $M_-^j(\alpha_r)$. In particular we have an isomorphism*
$$q_1 \colon q_1^{-1}(M_-^i(\alpha_r)) \cong M_-^i(\alpha_r).$$
(2) *The morphism $q_2 \colon M_-(\alpha_r, i) \to M_-(\alpha_{r-i})$ is a $Gr(n+1-r+i, i)$-bundle. In particular, we have an isomorphism $q_2 \colon M_-(\alpha_{n+1}, i) \cong M_-(\alpha_{n+1-i})$.*
(3) *For any $j \geq 0$, we have $q_1^{-1}(M_-^{i+j}(\alpha_r)) \cong q_2^{-1}(M_-^j(\alpha_{r-i}))$.*
(4) *The morphism $q_1' \colon M_+(\alpha_r, i) \to M_+(\alpha_r)$ is a $Gr(j,i)$-bundle over each strata $M_+^j(\alpha_r)$. In particular we have an isomorphism*
$$(q_1')^{-1}(M_+^j(\alpha_r)) \cong M_+^j(\alpha_r).$$
(5) *The morphism $q_2' \colon M_+(\alpha_r, i) \to M_+(\alpha_{r-i})$ is a $Gr(n-2-r+i, i)$-bundle. In particular we have an isomorphism $q_2' \colon M_+(\alpha_{n-2}, i) \cong M_+(\alpha_{n-2-i})$.*
(6) *For any $j \geq 0$, we have ${q_1'}^{-1}(M_+^{i+j}(\alpha_r)) \cong {q_2'}^{-1}(M_+^j(\alpha_{r-i}))$.*

Proof. (1) The fiber of q_1 over $E_- \in M_-^j(\alpha_r)$ is parametrized by $Gr(\mathrm{Hom}_B(S_0, E_-), i)$ for all $j \geq i$. For the universal bundle $\mathcal{U}$ in (6), as in § 3.3 we put $\delta^* := \sum_i \delta_i^* \otimes x_i \colon \mathcal{U}_0 \to \mathcal{U}_1 \otimes \mathbf{x}$. Then for any point $p \in M_-(\alpha_r)$, we have $\mathrm{Hom}_B(S_0, \mathcal{U}_p) \cong (\ker \delta^*)_p$. Since $\ker \delta^*$ is locally free of rank j on $M_-^j(\alpha_r)$ (cf. [ACGH84, Chapter II]), we have $Gr(j,i)$-bundle $Gr(\ker \delta^*|_{M_-^j(\alpha_r)}, i)$ on $M_-^j(\alpha_r)$.

On the other hand, by the definition of $M_-^j(\alpha_r)$ (cf § 3.2), we easily see that $M_-^j(\alpha_r)$ represents the moduli functor parametrizing families of θ_--semistable B-modules E_- with $[E_-] = \alpha_r$ and $\dim_{\mathbb{C}} \mathrm{Hom}_B(S_0, E_-) = j$. Hence $q_1^{-1}(M_-^j(\alpha_r))$ have the same universal property of $Gr(\ker \gamma^*|_{M_-^j(\alpha_r)}, i)$ and we have $q_1^{-1}(M_-^j(\alpha_r)) \cong Gr(\ker \gamma^*|_{M_-^j(\alpha_r)}, i)$.
(2) The fiber of q_2 over $E' = q_2(\bar{E}_-)$ is parametrized by $Gr\left(\mathrm{Ext}^1_B(E', S_0), i\right)$. For the universal family $\mathcal{U}' = \left(\mathcal{U}'_{-1} \overset{{\gamma'}_i^*}{\to} \mathcal{U}'_0 \overset{{\delta'}_j^*}{\to} \mathcal{U}'_1\right)$ of B-modules on $M_-(\alpha_{r-i})$, we put

$$\gamma'^* := \sum_i {\gamma'}_i^* \otimes x_i^* \colon \mathcal{U}_{-1} \otimes \mathbf{x} \to \mathcal{U}_0.$$

Since we have $(\ker \gamma'^*)^\vee_{p'} \cong \mathrm{Ext}^1_B(\mathcal{U}'_{p'}, S_0)$ for any $p' \in M_-(\alpha_{r-i})$. Similarly as in (1) we get

$$M_-(\alpha_r, i) \cong Gr\left((\ker \gamma'^*)^\vee, i\right).$$

(3) Since spaces of both sides have the same universal property, our claim holds.
(4), (5) and (6) are proved similarly as in (1), (2) and (3). $\square$

Corollary 3.10. *$M_-^i(\alpha_r)$ and $M_+^i(\alpha_r)$ are smooth for any $i, r \geq 0$.*

Proof. The restriction of the morphism $q_1 \colon M_-(\alpha_r, i) \to M_-(\alpha_r)$ gives an isomorphism

$$q_1^{-1}(M_-^i(\alpha_r)) \cong M_-^i(\alpha_r).$$

By Proposition 3.9 (3) we have an isomorphism $M_-^i(\alpha_r) \cong q_2^{-1}(M_-^0(\alpha_{r-i}))$. Hence by Proposition 3.9 (2), $M_-^i(\alpha_r)$ is isomorphic to a Grassmann-bundle over $M_-^0(\alpha_{r-i})$. Since $M_-^0(\alpha_{r-i})$ is smooth by Proposition 3.7, we see that $M_-^i(\alpha_r)$ is also smooth. Similarly $M_+^i(\alpha_r)$ is shown to be smooth. $\square$

3.6. **Stratified Grassmann bundle.** In this section we show that morphisms $f_\pm \colon M_\pm(\alpha_r) \to M_0(\alpha_r)$ are described by stratified Grassmann bundles using Proposition 3.9.

We consider the diagram:

$$\begin{tikzcd}
 & M_-(\alpha_{n+1}, n+1-r) \arrow[dl, "\cong"', "q_2"] \arrow[dr, "q_1"] & \\
M_-(\alpha_r) & & M_-(\alpha_{n+1}).
\end{tikzcd}$$

By Proposition 3.9 (2), q_2 is an isomorphism and we have a map $q_1 \circ q_2^{-1} \colon M_-(\alpha_r) \to M_-(\alpha_{n+1})$, which coincides with g_- by (15). This gives another proof of Theorem 1.1. Similarly the map $q_1' \circ {q_2'}^{-1} \colon M_+(\alpha_r) \cong M_+(\alpha_{n-2}, n-2-r) \to M_+(\alpha_{n-2})$ coincide with the map g_+ by (16). For any $r \geq 0$, we have isomorphisms $M_-^0(\alpha_r) \cong M_-^{n+1-r}(\alpha_{n+1})$ and $M_+^0(\alpha_r) \cong M_+^{n-2-r}(\alpha_{n-2})$ via g_- and g_+ respectively. In particular, we have isomorphisms

$$M_-^0(\alpha_{r-i}) \cong M_-^{n+1-r+i}(\alpha_{n+1}), M_+^0(\alpha_{r-i}) \cong M_+^{n-2-r+i}(\alpha_{n-2}).$$

By the diagram (14), $M_-^{n+1-r+i}(\alpha_{n+1})$ and $M_-^{n-2-r+i}(\alpha_{n-2})$ coincide with images $f_-(M_-^i(\alpha_r)) \cong f_+(M_+^i(\alpha_r))$ of f_- and f_+. This gives a proof of (2) in Main Theorem 1.4.

By Proposition 3.9 and the diagram (14) we also have proofs of (3) and (4) in Main Theorem 1.4.

3.7. **Hodge polynomials of flips.** We study the difference between Hodge polynomials of $M_-(\alpha_r)$ and $M_+(\alpha_r)$. To do this we use the virtual Hodge polynomial $e(Y) := \sum_{p,q} e^{p,q}(Y)x^p y^q$ for any variety Y (cf. [DK87]).

By Main Thoerem 1.4 we get the following diagram

$$\begin{tikzcd} \sqcup M_+^i(\alpha_r) \arrow[dr, "f_+"] & & \sqcup M_-^i(\alpha_r) \arrow[dl, "f_-"'] \\ & \sqcup M_0^i(\alpha_r) & \end{tikzcd} \tag{18}$$

where restrictions of f_+ and f_- to $M_\pm^i$ are $Gr(n-2-r+i,i)$-bundle and $Gr(n+1-r+i,i)$-bundle over $M_0^i(\alpha_r) \cong M_\pm^0(\alpha_{r-i})$, respectively. Hence we get the following equality.

$$\begin{aligned} &e\left(M_-(\alpha_r)\right) - e\left(M_+(\alpha_r)\right) = \\ &\qquad \sum_{i>0} \Big(e\left(Gr(n+1-r+i,i)\right) - e\left(Gr(n-2-r+i,i)\right) \Big) e\left(M_-^0(\alpha_{r-i})\right). \end{aligned} \tag{19}$$

In the following we compute the Hodge polynomial of $M_+(\alpha_r)$ from that of $M_-(\alpha_r)$ in the case where $r = 1, 2$. In this case, we know the Hodge polynomial of $M_-(\alpha_r) \cong M_{\mathbb{P}^2}(r,1,n)$ from [ES93] and [Yos94]. We need the following proposition.

Proposition 3.11. *We have following isomorphisms:*

$$M_-(\alpha_0) \cong M_+(\alpha_0) \cong \mathbb{P}^2.$$

A proof of this proposition is given in Appendix. From this proposition and (19), we get the following:

$$\begin{aligned} \cdots = M_+(1,1,1) &= M_+(1,1,2) = \emptyset, \\ e\left(M_+(1,1,3)\right) &= t^{12} + t^{10} + 3t^8 + 3t^6 + 3t^4 + t^2 + 1, \\ e\left(M_+(1,1,4)\right) &= t^{16} + 2t^{14} + 5t^{12} + 8t^{10} + 10t^8 + 8t^6 + 5t^4 + 2t^2 + 1, \\ e\left(M_+(1,1,5)\right) &= \cdots + 21t^{10} + 19t^8 + 11t^6 + 6t^4 + 2t^2 + 1, \\ e\left(M_+(1,1,n)\right) &= e\left(M_{\mathbb{P}^2}(1,1,n)\right) - (t^{2n+4} + 2t^{2n+2} + 3t^{2n} + 2t^{2n-2} + t^{2n-4}), \end{aligned}$$

and

$$\begin{aligned} \cdots = M_+(2,1,1) &= M_+(2,1,2) = M_+(2,1,3) = \emptyset, \\ e\left(M_+(2,1,4)\right) &= \cdots + 12t^{12} + 10t^{10} + 8t^8 + 5t^6 + 3t^4 + t^2 + 1, \\ e\left(M_+(2,1,5)\right) &= \cdots + 67t^{16} + 60t^{14} + 48t^{12} + 32t^{10} + 20t^8 + 10t^6 \\ &\quad + 5t^4 + 2t^2 + 1, \end{aligned}$$

where $t = xy$.

Appendix A. Proof of Proposition 3.11

We take $\alpha_0 \in K(\mathbb{P}^2)$ such that $\mathrm{ch}(\alpha_0) = -(0,1,n)$ as in §3 and give a proof of Proposition 3.11.

A.1. **Bridgeland stability.** We briefly introduce the concept of Bridgeland stability. For details the reader can consult [Bri07]. Let $\mathcal{A}$ be an abelian category, $K(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$.

Definition A.1. *A stability function Z on $\mathcal{A}$ is a group homomorphism from $K(\mathcal{A})$ to $\mathbb{C}$ satisfying that for any object $E \in \mathcal{A}$, if E is not equal to zero we have $Z(E) \in \mathbb{R}_{>0}\exp(\sqrt{-1}\pi\phi(E))$ with $0 < \phi(E) \le 1$.*

The real number $\phi(E)$ is called phase of E.

Definition A.2. *A nonzero object $E \in \mathcal{A}$ is semistable with respect to Z if and only if for any proper subobject $0 \neq F \subsetneq E$ we have $\phi(F) \leq \phi(E)$. If the inequality is always strict we call E to be stable with respect to Z.*

Let $\mathcal{T}$ be a triangulated category, $K(\mathcal{T})$ the Grothendieck group of $\mathcal{T}$.

Definition A.3. *A stability condition σ on $D^b(\mathbb{P})$ is a pair $\sigma = (\mathcal{A}, Z)$, which consists of a full subcategory $\mathcal{A}$ of $\mathcal{T}$ and a group homomorphism $Z \colon K(\mathcal{T}) \to \mathbb{C}$ satisfying following conditions:*

- *$\mathcal{A}$ is a heart of a bounded t-structure of $\mathcal{T}$, which implies $\mathcal{A}$ is an abelian category and $K(\mathcal{A})$ is isomorphic to $K(\mathcal{T})$ by the inclusion $\mathcal{A} \subset \mathcal{T}$. Hence we always identify them.*
- *Z is a stability function on $\mathcal{A}$ via the above identification $K(\mathcal{A}) = K(\mathcal{T})$.*
- *Z has Harder-Narasimhan property.*

We omit the definition of “a heart of a bounded t-structure” and “Harder-Narasimhan property” (see [Bri07, § 2 and § 3]). We denote a set of all stability conditions satisfying a technical condition called “local finiteness” (see [Bri07, § 5]) by $\mathrm{Stab}(\mathcal{T})$.

Definition A.4. *For a stability condition $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}(\mathcal{T})$, an object $E \in \mathcal{T}$ is called σ-(semi)stable if and only if E belongs to $\mathcal{A}$ up to shift functors $[n] \colon \mathcal{T} \to \mathcal{T}$ for $n \in \mathbb{Z}$, and it is semistable with respect to Z.*

In the following we only consider the case where $\mathcal{T} = D^b(\mathbb{P}^2)$ and we put $\mathrm{Stab}(\mathbb{P}^2) := \mathrm{Stab}(\mathcal{T})$. For $\alpha \in K(\mathbb{P}^2)$ and $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}(\mathbb{P}^2)$, we define a moduli functor $\mathcal{M}_{D^b(\mathbb{P}^2)}(\alpha, \sigma)$ of σ-semistable objects $E \in \mathcal{A}$ with $[E] = \alpha \in K(\mathbb{P}^2)$ as follows. The moduli functor $\mathcal{M}_{D^b(\mathbb{P}^2)}(\alpha, \sigma)$ is a functor from $(\mathrm{Sch}/\mathbb{C})$ to (Set). For a scheme S over $\mathbb{C}$ it sends S to a set $\mathcal{M}_{D^b(\mathbb{P}^2)}(\alpha, \sigma)(S)$ of families $\mathcal{F} \in D^b(\mathbb{P}^2 \times S)$ of σ-semistable objects with class α in $K(\mathbb{P}^2)$. This means that for any $\mathbb{C}$-valued point $s \in S$, the fiber $\mathbf{L}\iota_s^* \mathcal{F} \in D^-(\mathbb{P}^2)$ belongs to the full subcategory $\mathcal{A} \subset D^b(\mathbb{P}^2)$ and σ-semistable with $[\mathbf{L}\iota_s^* \mathcal{F}] = \alpha \in K(\mathbb{P}^2)$.

There exists a right action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on $\mathrm{Stab}(\mathbb{P}^2)$ and this action does not change semistable objects. Hence for any $\alpha \in K(\mathbb{P}^2)$, $\sigma \in \mathrm{Stab}(\mathbb{P}^2)$ and $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, there exists an integer $n \in \mathbb{Z}$ such that shift [n] induces an isomorphism of functors

$$\mathcal{M}_{D^b(\mathbb{P}^2)}(\alpha, \sigma) \cong \mathcal{M}_{D^b(\mathbb{P}^2)}((-1)^n \alpha, \sigma g) \colon E \mapsto E[n].$$

A.2. **Geometric stability.** Let H be the ample generator of $\mathrm{Pic}(\mathbb{P}^2)$ and $s, t \in \mathbb{R}$ with $t > 0$. For any torsion free sheaf E on $\mathbb{P}^2$, the slope of E is defined by $\mu_H(E) := \frac{c_1(E)}{\mathrm{rk}(E)}$ and define μ_H-semistability. E has the Harder-Narasimhan filtration with μ_H-semistable factors. We denote the maximal value and the minimal value of slopes of μ_H-semistable factors of E by $\mu_{H-\max}(E)$ and $\mu_{H-\min}(E)$, respectively. Then we define a pair $\sigma_{(sH,tH)} = (\mathcal{A}_{(sH,tH)}, Z_{(sH,tH)})$ as follows.

Definition A.5. *An object $E \in D^b(\mathbb{P}^2)$ belongs to the full subcategory $\mathcal{A}_{(sH,tH)}$ if and only if*

- *$\mathcal{H}^i(E) = 0$ for all $i \neq 0, -1$*
- *$\mathcal{H}^0(E)$ is torsion or $\mu_{H-\min}(\mathcal{H}^0(E)_{\mathrm{fr}}) > st$, where $\mathcal{H}^0(E)_{\mathrm{fr}}$ is the free part of $\mathcal{H}^0(E)$*

- *$\mathcal{H}^{-1}(E)$ is torsion free and $\mu_{H-\max}(\mathcal{H}^{-1}(E)) \leq st$.*

The group homomorphism $Z_{(sH,tH)}$ is defined by

$$Z_{(sH,tH)}(E) := -\int_{\mathbb{P}^2} \mathrm{ch}(E) \exp(-sH - \sqrt{-1}tH).$$

If s and t belong to $\mathbb{Q}$, then $\sigma_{(sH,tH)}$ is a stability condition on $D^b(\mathbb{P}^2)$ (cf. [ABL]). In general we do not know wheather $\sigma_{(sH,tH)}$ is a stability condition on $D^b(\mathbb{P}^2)$. We have the following criterion due to Bridgeland.

Proposition A.6. cf. [Ohk, Proposition 3.6] *For $\sigma \in \mathrm{Stab}(\mathbb{P}^2)$, there exist $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ and $s, t \in \mathbb{R}$ with $t > 0$ such that $\sigma = \sigma_{(sH,tH)}g$ if and only if the following conditions* (i) *and* (ii) *hold.*

(i) *For any closed point $x \in \mathbb{P}^2$, the skyscraper sheaf $\mathcal{O}_x$ is σ-stable.*
(ii) *For any $\beta \in K(\mathbb{P}^2)$, if $Z(\beta) = 0$ then $c_1^2 - 2r\,\mathrm{ch}_2 < 0$ where $\mathrm{ch}(\beta) = (r, c_1, \mathrm{ch}_2)$.*

A.3. **Proof of Proposition 3.11.** We take $\sigma^s = (\mathcal{A}, Z^s) \in \mathrm{Stab}(\mathbb{P}^2)$ for $s \in \mathbb{R}$ with $-1 < s < 1$, where

$$\mathcal{A} = \langle \mathcal{O}_{\mathbb{P}^2}(-1)[2], \mathcal{O}_{\mathbb{P}^2}[1], \mathcal{O}_{\mathbb{P}^2} \rangle$$

and Z^s is a group homomorphism $Z^s \colon K(\mathbb{P}^2) \to \mathbb{C}$ defined by

$$Z^s(e_{-1}) = \frac{-s-1}{2}, Z^s(e_0) = 1 + \sqrt{-1}, Z^s(e_1) = \frac{-s+1}{2}$$

for $e_i = [\mathcal{O}_{\mathbb{P}^2}(i)[1-i]] \in K(\mathbb{P}^2)$, $i = -1, 0, 1$. Then by Proposition A.6 we see that there exists an element $g^s \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that

$$\sigma^s = \sigma_{(sH,tH)} g^s, \tag{20}$$

where $t = \sqrt{1 - s^2}$.

We take $\alpha_0 = {}^t(n, 2n-1, n-1) \in K(B)$ and define a group homomorphism $\tilde{\theta}^s \colon K(B) \to \mathbb{C}$ by

$$\tilde{\theta}^s(\beta) = \det \begin{pmatrix} \mathrm{Re}\, Z^s(\beta) & \mathrm{Re}\, Z^s(\alpha_0) \\ \mathrm{Im}\, Z^s(\beta) & \mathrm{Im}\, Z^s(\alpha_0) \end{pmatrix}$$

for each $\beta \in K(B)$. Then by [Ohk, Proposition 1.2], $M_B(\alpha_0, \tilde{\theta}^s)$ corepresents the moduli functor $\mathcal{M}_{D^b(\mathbb{P}^2)}(\alpha_0, \sigma^s)$. Furthermore by (20), we have an isomorphism

$$\mathcal{M}_{D^b(\mathbb{P}^2)}(-\alpha_0, \sigma_{(sH,tH)}) \cong \mathcal{M}_{D^b(\mathbb{P}^2)}(\alpha_0, \sigma^s) \colon E \mapsto E[1] \tag{21}$$

of moduli functors. We notice that for an object $E \in \mathcal{A}_{(sH,tH)}$, we have $[E] = -\alpha_0 \in K(B) \cong K(\mathbb{P}^2)$ if and only if $\mathrm{ch}(E) = (0, 1, \frac{1}{2} - n) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}$. We give the following lemmas to prove Proposition 3.11

Lemma A.7. *We assume $-1 < s \leq 0$ and put $t = \sqrt{1 - s^2}$. Then for any line $L \subset \mathbb{P}^2$, the structure sheaf $\mathcal{O}_L(1-n)$ tensored by $\mathcal{O}_{\mathbb{P}^2}((1-n)H)$ is $\sigma_{(sH,tH)}$-stable.*

Proof. We show that $\mathcal{O}_L(1-n) \in \mathcal{A}_{(sH,tH)}$ is $\sigma_{(sH,tH)}$-stable for any line $L \subset \mathbb{P}^2$. We take an exact sequence in $\mathcal{A}_{(sH,tH)}$

$$0 \to F \to \mathcal{O}_L(1-n) \to G \to 0.$$

Then we have a long exact sequence

$$0 \to \mathcal{H}^{-1}(G) \to F \to \mathcal{O}_L(1-n) \to \mathcal{H}^0(G) \to 0.$$

If the dimension of support of $\mathcal{H}^0(G)$ is equal to 1, we have $\mathrm{rk}(F) = \mathrm{rk}(\mathcal{H}^{-1}(G))$, $c_1(F) = c_1(\mathcal{H}^{-1}(G))$. If $\mathrm{rk}(F) \neq 0$, this contradicts the fact that $F, G \in \mathcal{A}_{(sH,tH)}$ implies inequalities $\mu_{tH}(\mathcal{H}^{-1}(G)) \leq st < \mu_{tH}(F)$. Hence F is a torsion sheaf and

$\mathcal{H}^{-1}(G) = 0$. This implies $F = 0$ and $G = \mathcal{O}_L(1-n)$ since $\mathcal{O}_L(1-n)$ is a pure sheaf.

If the dimension of support of $\mathcal{H}^0(G)$ is equal to 0, we have $\mathrm{rk}(F) = \mathrm{rk}(\mathcal{H}^{-1}(G))$, $c_1(F) = c_1(\mathcal{H}^{-1}(G)) - 1$. Inequalities $\mu_H(\mathcal{H}^{-1}(G)) \le st < \mu_H(F)$ implis

$$c_1(F) = 1, \text{ and } c_1(\mathcal{H}^{-1}(G)) = 0.$$

Hence we have $\mathrm{Im}\, Z_{(sH,tH)}(G) = \mathrm{rk}(\mathcal{H}^{-1}(G))st$. In the case where $s < 0$ this implies $\mathcal{H}^{-1}(G) = 0$ since $\mathrm{Im}\, Z_{(sH,tH)}(G) \ge 0$. In any case, we have $\mathrm{Im}\, Z_{(sH,tH)}(G) = 0$ and

$$\phi(G) = 1 > \phi(\mathcal{O}_L(1-n)).$$

Hence G does not break $\sigma_{(sH,tH)}$-stability of $\mathcal{O}_L(1-n)$. □

Lemma A.8. *An object $E \in \mathcal{A}_{(0,H)}$ with $[E] = -\alpha_0$ is $\sigma_{(0,H)}$-semistable if and only if $E \cong \mathcal{O}_L(1-n)$ for a line L on $\mathbb{P}^2$.*

Proof. We assume that $E \in \mathcal{A}_{(0,H)}$ is $\sigma_{(0,H)}$-semistable and $\mathcal{H}^{-1}(E) \neq 0$. We put $F := \mathcal{H}^{-1}(E)[1]$ and $G := \mathcal{H}^0(E)$. Then an exact sequence in $\mathcal{A}_{(0,H)}$

$$0 \to F \to E \to G \to 0$$

implies that $0 \le \mathrm{Im}\, Z_{(0,H)}(F) < \mathrm{Im}\, Z_{(0,H)}(E) = t \le 1$. If we put $\mathrm{ch}(F) = -(r, c_1, \mathrm{ch}_2)$ with $r > 0$. Then $\mathrm{Im}\, Z_{(0,H)}(F) = -c_1 = 0$ hence we see that $\phi(E) < \phi(F) = 1$ contradicting to $\sigma_{(0,H)}$-semistability of E. Thus E is a sheaf with $\mathrm{ch}(E) = (0, 1, \frac{1}{2} - n)$. Any subsheaf with support dimension 1 of E break $\sigma_{(0,H)}$-semistability of E. Hence we see that E is a pure sheaf. This shows that $E \cong \mathcal{O}_L(1-n)$ for a line L on $\mathbb{P}^2$.

Conversely by Lemma A.7, we see that $\mathcal{O}_L(1-n) \in \mathcal{A}_{(0,H)}$ is $\sigma_{(0,H)}$-semistable for any line $L \subset \mathbb{P}^2$. □

By this lemma and the isomorphism (21), the moduli functor $\mathcal{M}_{D^b(\mathbb{P}^2)}(\alpha_0, \sigma^0)$ is represented by $\mathbb{P}^2 \cong \{\mathcal{O}_L(1-n) \mid L \subset \mathbb{P}^2\colon \text{ line }\}$. By [Ohk, Proposition 4.4], $\mathcal{M}_{D^b(\mathbb{P}^2)}(\alpha_0, \sigma^0)$ is also represented by $M_B(\alpha_0, \tilde{\theta}^0)$. Hence we have an isomorphism $M_B(\alpha_0, \tilde{\theta}^0) \cong \mathbb{P}^2$. If we put $s_0 := -\frac{1}{2n-1}$, then we have $\tilde{\theta}^{s_0+\varepsilon} \in C_-$, $\tilde{\theta}^{s_0-\varepsilon} \in C_+$ and $\tilde{\theta}^{s_0} \in W_0$ for $\varepsilon > 0$ small enough. By Lemma A.7 every object in $M_B(\alpha_0, \tilde{\theta}^0)$ is $\tilde{\theta}^s$-stable for $-1 < s \le 0$. Hence W_0 is not a wall and $C_\pm$ and W_0 are contained in a single chamber. As a consequence we have isomorphisms

$$M_+(\alpha_0) \cong M_-(\alpha_0) \cong M_B(\alpha_0, \tilde{\theta}^0) \cong \mathbb{P}^2.$$

This completes the proof of Proposition 3.11.

Acknowledgement

The author is grateful to his adviser Takao Fujita for many valuable comments and encouragement. He thanks Hiraku Nakajima and Kōta Yoshioka for valuable comments and motivating him to write this paper.

References

[ABL] D. Arcara, A. Bertram and M. Lieblich. Bridgeland-Stable Moduli Spaces for K-Trivial Surfaces. math.AG/0708.2247.

[ACGH84] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris. *Geometry of Algebraic curves*, Vol. I. Springer, 1984.

[BCHM10] C. Birkar, P. Cascini, C. Hacon and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, Vol. 23, pp. 405-468, 2010.

[Bon89] A. I. Bondal. Representations of associative algebras and coherent sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.*, Vol. 53, pp. 25–44, 1989. English transl. in Math. USSR-Izv. 34 (1990), no. 1, 23-44.

[Bri07] T. Bridgeland. Stability conditions on triangulated categories. *Ann. of Math.*, Vol. 166, No. (2), pp. 317–345, 2007.

[DK87] V. I. Danilov and A. G. Khovanskii. Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers. *Math. USSR Izvestiya*, Vol. 29, pp. 279–298, 1987.

[Dre98] J. M. Drezet. Group de Picard des variétés de modules de faisceaux semi-stables sur $\mathbb{P}_2(\mathbb{C})$. *Ann. de l'Institut Fourier.*, Vol. 38, pp. 105–168, 1998.

[ES93] G. Ellingsrud and S. A. Strømme. Towards the Chow ring of the Hilbert scheme on $\mathbb{P}^2$. *J. Reine Angew. Math.*, Vol. 441, pp. 33–44, 1993.

[HL97] D. Huybrechts and M. Lehn. *Geometry of moduli spaces of sheaves.* No. E31 in Aspects in Mathematics. Vieweg, 1997.

[Li97] J. Li. *Compactification of moduli of vector bundles over algebraic surfaces.* World. Sci. Publishing, River Edge, NJ, 1997.

[Mar88] M. Maruyama. *On a compactification of a moduli space of stable vector bundles on a rational surface.* Kinokuniya, Tokyo, 1988.

[NY] H. Nakajima and K. Yoshioka. Perverse coherent sheaves on blow-up. II. wall-crossing and Betti numbers formula. arXiv:0806.0463.

[Ohk] R. Ohkawa. Moduli of Bridgeland semistable objects on $\mathbb{P}^2$. To appear in Kodai Mathematical Journal.

[Pot94] Le Potier. À propos de la construction de l'espace de modules des faisceaux semi-stables sur le plan projectif. *Bull. Soc. Math. France*, Vol. 122, pp. 363–369, 1994.

[Pot97] Le Potier. *Lectures on vector bundles.* No. 54 in Cambridge Stud. Adv. Math. Cambridge University Press, Cambridge, 1997.

[Yos94] K. Yoshioka. The Betti numbers of the moduli space of stable sheaves of rank 2 on $\mathbb{P}^2$. *J. Reine Angew, Math.*, Vol. 453, pp. 193–220, 1994.

[Yos03] K. Yoshioka. A note on moduli of vector bundles on rational surfaces. *J. Math. Kyoto Univ.*, Vol. 43, pp. 139–163, 2003.

Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, JAPAN

E-mail address: ookawa@math.titech.ac.jp